RESEARCH STATEMENT

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My research interests lie at the intersection of geometric topology, homotopy theory, and functional analysis. Specifically, I aim to study problems related to characteristic classes and other invariants of families of manifolds, or manifold bundles, using a combination of homotopy theoretic and functional analytic methods, in particular utilizing index theoretic interpretations of these invariants.

By a manifold bundle we mean a family of manifolds parametrized by a "nice" topological space. More precisely, by an M-bundle we mean a fiber bundle $E \to B$ with fiber a smooth manifold M and structure group the group of diffeomorphisms $Diff(M)$ of M. General knowledge of the ring $H^*(BDiff(M))$, for H^* a generalized cohomology theory (see [1]), better known as characteristic classes of M-bundles, is particularly limited. The solutions to Mumford's conjecture by Madsen and Weiss [24] and by Galatius, Madsen, Tillman, and Weiss [20] highlighted a universal perspective from which to consider a large family of characteristic classes of manifold bundles, known as tautological classes. This universal perspective, and applications of the techniques it has inspired, is the theme underlying most of my research. Many of the constructions involved in this perspective admit interpretations in terms of the index theory of elliptic operators. Characteristic of these interpretations is a unique interplay between modern homotopy theoretic ideas and well worn classical ideas from bordism and index theory. Broadly speaking, the goal of my research is to further explore and exploit this interplay.

This statement is divided in to 3 sections. The first two essentially summarize the authors thesis work on characteristic classes of M-bundles. In section 1 we give some background on manifolds bundles and tautological classes necessary to motivate and state our main results. The results in section 1 were obtained by considering the index of a certain family of elliptic operators, the odd signature operator D^o_V with coeffiecients in a flat bundle. In section 2 we give a brief outline of how characteristic classes of M-bundles can be realized as indices of elliptic operators and state our vanishing theorem for D^o_V. The results of section 1 and 2 generalize the main results of [14] which provided the main motivation for our thesis work. Section 3 consists of an outline of some possible directions for future research.

1. Manifold Bundles and Characteristic classes

Let $f : E \to B$ be smooth M-bundle, that is a smooth fiber bundle with fiber M a smooth closed oriented Riemannian manifold. Recall that isomorphism classes of M-bundles over a nice space B correspond to homotopy classes of maps $c_E : B \to BDiff(M)$ from the base space B to the classifying space $BDiff(M)$. Thus we may view $H^*(BDiff(M))$ as the universal home of characteristic classes of M-bundles. These are sent to elements in $H^*(B)$ under the induced map $c_E^* : H^*(BDiff(M)) \to H^*(B)$.

Unlike $H^*(BG)$ with G a classical linear group, $H^*(BDiff(M))$ is notoriously difficult to understand. This has caused the study of characteristic classes of smooth manifold bundles to lack much of a unifying theme, progress being made only a space at a time in some sense (e.g. cirlce bundles, surface bundles, and some 3-manifolds).

There are however tautological classes, also know as generalized Mumford-Morita-Miller classes, defined for all manifold bundles. These classes can be studied more holistically in the sense that statements may be obtained that are true for large families of manifolds (or manifold bundles), usually only dependent on dimensional constraints, for example the main results of [17], [32]. Following the resolution of Mumfords conjecture there has been a surge of activity related to these classes (see [14], [13], [21], [17], [32], [19], [12]). Our graduate research focused on extending the main techniques used in [14] to prove statements about more general invariants. We'll now provide some background material before stating some of our recent results.

Let $MTSO(n)$ denote the Thom spectrum of the (additive) inverse of the universal n-dimensional oriented vector bundle over $BSO(n)$. Given an oriented closed manifold M there exist a map

$$\alpha : \Sigma^\infty(BDiff^+(M)_+) \to MTSO(n)$$

coming from the parametrized Pontrjagin–Thom collapse (see [20], [18], or [19]). Under the induced map

$$\alpha^* : H^*(MTSO(n)) \to H^*(BDiff^+(M)),$$

every element of $H^*(MTSO(n))$ produces an element in $H^*(BDiff(M))$, i.e. a characteristic class of M-bundles. Given a M-bundle $E \to B$, the map α may be composed with the classifying map $c_E : B \to BDiff(M)$ to produce the so-called *Madsen-Tillman-Weiss* map (**MTW**-map for short):

$$\alpha_E := \alpha \circ c_E : B \to MTSO(n)$$

of the M-bundle $E \to B$. The **tautological characteristic classes** of the bundle $E \to B$ may be described as

$$\alpha_E^*(c) \in H^*(B)$$

for $c \in H^*(MTSO(n))$, where $\alpha_E^* : H^*(MTSO(n)) \to H^*(B)$ is the induced map.

Understanding these classes is greatly simplified by the fact that, for many theories H^*, $H^*(MTSO(n))$ is well understood and expressible in terms of $H^*(BSO(n))$ via the Thom isomorphism. In particular, $H^*(MTSO(n), \mathbb{Q}) \cong H^{*+n}(BSO(n), \mathbb{Q})$. The study the MTW-map has led to many interesting results about the tautological characteristic classes of M-bundles and other related invariants, for example [14], [13], [21], [17], [32], [19], [12], and the authors thesis, to name a few.

In [13] Ebert answered perhaps the most begging question related to the tautological classes, the detection question. That is, given a class $c \in H^*(MTSO(n), \mathbb{Q})$, is there a M-bundle $E \to B$ for which $\alpha_E^*(c) \in H^*(B, \mathbb{Q})$ is nontrivial? For $n = 2m$ even the answer is always yes, however this is not always so for $n = 2m - 1$ odd. His results can be interpreted ([14] corollary 1.4) as identifying the signature of a manifold as an obstruction to a space fibering as an M-bundle with M odd dimensional. Although this had been known for some time (proved in [23] and [26]), this interpretation motivates the generalizations of the main results of [14] proven in the authors thesis which identifies the higher signatures, defined below, as obstructions to fibering as an odd dimensional manifold bundle.

For a manifold M with $u : M \to B\pi$ a map classifying its universal cover, $\pi = \pi_1(M)$, the **higher signatures of** M are characteristic numbers of the form

$$\langle L(M) \cup u^*(x), [M] \rangle$$

where $x \in H^*(B\pi, \mathbb{Q})$ and $L = \Sigma\ L_i \in H^{4*}(BSO; \mathbb{Q})$ is the (modified) total Hirzebruch L-class associated with the formal power series $\frac{\sqrt{x}/2}{tanh(\sqrt{x}/2)}$ (see [27] chapter 19). These characteristic numbers where conjectured by S.P. Novikov [31] to be invariants of the oriented homotopy type of M.

Recall that a vector bundle $V \to M$ is *flat* if V admits a connection with vanishing curvature. Equivalently, V is flat if $V \cong u^* V_\rho$, where $V_\rho \to B\pi$ is the vector bundle over $B\pi$ induced by a finite dimensional representation ρ of π. Let $ch : K \to H\mathbb{Q}$ denote the chern character. The higher signatures of the form $\langle L(M)ch([V]), [M] \rangle$ with V flat are related to an intersection form defined on the *cohomology* $H^*(M; V)$ *of* M *with coefficients in* V (just as for Hirzebruch's signature $Sig(M) = \langle L(M), [M] \rangle$). See [28] for definitions and more on higher signatures. For $V \to M$ flat, $H^*(M; V)$ are finite dimensional vector spaces which can be identified, using general hodge theory as in [2], [3], as the kernel of certain elliptic operators. It is this observation which is crucial for the proof of theorem 2.1 (stated in the next section), from which the following results (theorems 1.1 and 1.2) are derived.

Theorem 1.1. *Suppose* $f : E \to B$ *is an oriented* M*-bundle, with* B *and* M *both oriented smooth closed manifolds. If the dimension of* M *is odd then the higher signatures of* E*,* $\langle L(E)ch([V]), [E] \rangle = 0$*, for all flat Hermitian vector bundles* $V \to E$*.*

Theorem 1.1 generalizes corollary 1.4 of [14] (where $V = 0$). Note the flatness of the vector bundle in theorem 1.1 is critical (i.e. it is false without). This can be seen with the simplest example $E = S^1 \times S^1$. From here on, when we refer to higher signatures we will mean those of the form $\langle L(M)ch([V]), [M] \rangle$ with $V \to M$ a flat Hermitian vector bundle.

Integration over the fiber of $f : E^{n+k} \to B^k$, aka the gysin map $f_! : H^*(E) \to H^{*-n}(B)$, may be used to express characteristic numbers of E as characteristic numbes of B:

$$\langle X, [E] \rangle = \langle f_!(X), [B] \rangle,$$

for any $X \in H^*(E)$.

Since the L-class is multiplicative and $TE = f^*TB + T_vE$, where $T_vE = ker(df)$ is the vertical tangent bundle, for $X = L(E)ch([V])$ this gives

$$\begin{aligned} \langle L(E)ch([V]), [E] \rangle &= \langle L(f^*TB)L(T_vE)ch([V]), [E] \rangle \\ &= \langle L(B)f_!(L(T_vE)ch([V])), [B] \rangle. \end{aligned}$$

Therefore theorem 1.1 is an immediate consequence of the following:

Theorem 1.2. *Suppose $f : E \to B$ is a smooth oriented M-bundle, M a smooth closed manifold. If the dimension of M is odd then $f_!(L(T_vE)ch([V])) = 0 \in H^*(B, \mathbb{Q})$ for all flat Hermitian vector bundles $V \to E$.*

Just as the special case ([14] theorem 1.2), the proposition is the cohomological consequence of the index of a certain family of elliptic operators vanishing, which will be explained in the next section.

Note the classes $f_!(\mathcal{L}(T_vE)ch([V]))$ are not tautological classes at all. Thus theorem 1.1 and 1.2 demonstrate how the universal perspective provided by GMTW context, and techniques used thus far to explore the tautological classes, can inform our understanding and ability to study more general characteristic classes and numbers associated to manifold bundles. Our strong suspicion is that more work of this nature is possible.

2. Elliptic Operators

Another source of characteristic classes of manifold bundles comes from the index theory of elliptic operators. An **elliptic operator** D on a smooth manifold M is a linear operator $D : \Gamma(V_0) \to \Gamma(V_1)$ acting between the spaces of sections $\Gamma(-)$ of vector bundles $V_0, V_1 \to M$ and satisfying certain local conditions. A **family** $D = \{D_b\}_{b\in B}$ **of elliptic operators** acting on the fibers of a M-bundle $f : E \to B$ is a collection of elliptic operators, parametrized by B, where each D_b is an elliptic operator on the fiber $f^{-1}(b) \cong M$ above $b \in B$. See [6] and [5] for full definitions.

The **index** of a family of operators D acting on the fibers of a M-bundle $f : E \to B$ is an element

$$ind(D) \in K^*(B)$$

in the complex K-theory of the base space. These may be passed through the Chern character to obtain rational cohomology classes. Many well known invariants (e.g. the Euler characteristic, (higher) signature, and A-hat genus) are expressible as indices of elliptic operators.

For example, the signature of a even dimensional manifold may be computed in terms of the index of the *even signature operator*, which acts between certain spaces of differential forms (see [5]). The even signature operator has an odd counterpart, first defined by Atiyah in [4]. The **odd signature operator** D^o is defined for all odd dimensional manifolds M^{2m-1} as follows. $D^o : A^{ev}(M) \to A^{ev}(M)$ acts between the space $A^{ev}(M) = \bigoplus_{k=p}^{m} A^{2p}(M)$ of even degree complex valued differential forms as

$$D\phi = i^m(-1)^{p+1}(d\star - \star d)\phi$$

for $\phi \in A^{2p}(M)$, where d is the exterior derivative and $\star$ denotes the Hodge star operator. Given a flat Hermitian vector bundle $V \to M$, we can "twist" D^o by V to obtain an operator $D^o_V : A^{ev}(M; V) \to A^{ev}(M; V)$ which now acts on even degree forms with *coefficients in the flat bundle V*.

The results of the last section were obtained by considering the family of odd signature operators with coefficients in a flat Hermitian bundle $V \to E$ acting on the fibers of an M-bundle $E \to B$:

Theorem 2.1. *Let $f : E \to B$ be an oriented M-bundle, M a closed oriented Riemannian manifold of odd dimension, $V \to E$ a flat Hermitian vector bundle. The the family index of the odd signature operator twisted by V is zero, $index(D^o_V) = 0 \in K^1(B)$*

The proof of theorem 2.1 is essentially functional analytic in nature. It relies on a supposedly well know fact in functional analysis stated with proof as theorem 4.1 in [14] where it is applied by Ebert to show the family index of the (untwisted) odd signature operator D^o is zero. Theorem 2.1 follows from the fact that the crucial observations made of D^o by Ebert in [14] are true more generally for D^o_V, thus theorem 4.1 of [14] can be applied to D^o_V to draw the same conclusion, in exactly the same manner, as was done for D^o in [14].

Again, although D^o_V is not symbolically universal in the sense of [14] section 3.2, the considerations made there greatly simplify the work need to derive theorem 1.1 and 1.2 above. In particular the calculation for the universal symbol class of the (untwisted) signature operators given as proposition 4.2 of [14] is essentially all we need.

3. Future Plans

3.1. **Manifold Bundles and Higher Signature.** Currently I am preparing my thesis for publication and would like to obtain twisted versions of some additional results from [14]. In particular, some extension of the real refinement (theorem 5.1 of [14]) should be possible. Removing certain, likely unnecessary, smoothness and compactness condition on the spaces involved in the above theorems is another a priority.

Continuing with the study of invariants of manifold bundles, it would be interesting to compare the vanishing results of [14] and the above generalizations with the vanishing conditions imposed by Bott's vanishing theorem [9]. Bott's vanishing theorem states that certain Pontrygin and Chern classes associated to a foliated manifold always vanish. Since a manifold bundle may be viewed as foliated by its fibers, many of the characteristic classes of the total space necessarily vanish. We would like to know how strong of a statement can be obtained about the kernel of the pushforward $f_! : H^*(E) \to H^{*-n}(B)$ and the L-classes of the total space of a manifold bundle $f : E \to B$ using Bott's vanishing theorems and other relatively elementary considerations. It seems likely that there are many cases when the L_i-class of E are already zero before being pushed forward by $f_!$.

We would also like to get some idea of how useful theorem 1.1 above really is as an obstruction to fibering,

Question: Can we find more manifolds with non-zero higher signatures?

The simplest examples of manifolds with non-zero higher signatures are certain surfaces described in [22] section 8. It is at least true that the higher signatures provide a more general obstruction than the usual signature since these are surfaces and therefore have zero signature. Due to formal properties, various products of these surfaces will have non-zero higher signatures as well. Perhaps a more pointed questions is

Question: Can we find more manifolds with non-zero higher signatures that aren't realizable as products of surfaces?

Some indication of how to approach this problem is given in [34] and [25]. Our hope is that combining the considerations made in [34] and [25] with those of [22] will lead to the desired examples.

The ideas discussed in this section so far are in some sense natural extensions of the ideas contained in the author's thesis. My plans moving forward fall roughly into two categories.

3.2. **Geometric Bordism Theory.** Firstly, I'd like to continue the revival of geometric aspects of cobordism theory eluded to in the introduction [15] and established in [14], [13], [21], [17], [32], [19], and [12], among others, by finding applications for the general index theorem proved by Ebert in [15] and studying tautological classes associated to more general tangential structures.

The connections between bordism and index theory demonstrated in [15] are reminiscent of the theme of [7]. In [7] Baum and Douglas provided a K-homological statement of the Atiyah-Singer index theorem (as opposed to the traditional K-*co*homological). The authors give two definitions of the cycles representing K-homology: one geometric/topological, denoted K_*^t, and one analytic, denoted K_*^a, and prove $K_*^t(X) \cong K_*^a(X)$ for all reasonable spaces X. Baum and Douglas' isomorphism provides a unified framework for treating many diverse index theorems. As Ebert's index theorem generalizes many of the theorems the Baum-Douglas framework was intended to treat it seems natural to ask

Question Can we put Ebert's index theorem into the Baum-Douglas context?

Of course, using the notation of [15], this would entail describing topological and analytic cycles for $\mathbb{K}A_*(X)$ and proving they determine the same theory. This is almost certainly possible, and likely not inordinately difficult, however we feel it would go a long way towards making the theory described in [15] more digestible for geometers and others who are not necessarily as well versed in homotopy theory.

Let $\theta : X \to BO(n)$ be a fibration and let $MT\theta(n)$ denote the Thom spectrum of the of the additive inverse of the vector bundle classified by θ. The MTW-map above is a special case of a more general map $\alpha : \Sigma^\infty BDiff_\theta(M)_+ \to MT\theta(n)$ defined for any θ-manifolds M (see [15] or [20] for details).

Let Γ_n denote Haefliger's Category. Γ_n is the topological category whose objects are points in $\mathbb{R}^n$ with the usual topology and morphisms between two points, say x and y, are germs of diffeomorphisms taking x to y. The dericative induces a continuous functor $\Gamma_n \to Gl_n(\mathbb{R})$ which induces a map $\nu_n : B\Gamma_n \to BGl_n(\mathbb{R})$. The space $B\Gamma_n$ classifies Γ_n-foliations (see [10], [8]) which are closely related to genuine foliations (in particular every foliation determines a Γ_n-foliation).

Question What can the MTW-map associated to the fibration $B\Gamma_n \to BGl_n(\mathbb{R}) \simeq BO(n)$ tell us about characteristic classes of foliated manifold bundles?

S. Nariman has shown this question is worth considering ([29] [30]). The homology of Haefliger's classifying space is still only poorly understood, however Nariman has been able to deduce a number of interesting results by considering diffeomorphism groups with the discrete topology and proving analogues of homological stability results proven by Galatius and Randal-Williams. We'd like to pursue this direction further with an eye towards index theoretic interpretations involving elliptic operators on foliated manifolds.

3.3. **Scalar Curvature.** Secondly, I'd like explore ties between my thesis work and the recent work of Ebert and Randal Williams in [16] and [11] which utilizes the MTW-map to study problems related to the existence of metrics with positive scalar curvature (hereafter, psc).

Among the several curvature conditions one can put on a Riemannian metric, the condition of positive scalar curvature provides perhaps the richest connection with topology, and in particular to cobordism theory. I'd like to use the connections provided in [16] and [11] between the Madsen-Tillman-Weiss spectra and the space of positive scalar curvature metrics to draw conclusions about the existence of psc metrics on the total space of a M-bundle.

Question: If all the higher signatures of a Riemannian manifold vanish, what conditions does this impose on curvature?

Novikov conjectured that the higher signatures are oriented homotopy invariants. This conjecture has motivated decades of research and developed deep connections with many important questions in topology, however the author is unaware of much specific geometric or topological content that can be derived from considering the higher signatures of a particular manifold. At least, by theorem 1.1, we know that the higher signature may be interpreted as obstructions to fibering. On the other hand, for a Riemannian spin manifold M which admits a psc metric, it is easy to show that $\widehat{\mathcal{A}}(M) = \langle \widehat{\mathcal{A}}(M), [M] \rangle = 0$, where $\widehat{\mathcal{A}}$ is the A-hat genus. In [33] J. Rosenberg conjectured that what he called the higher $\widehat{A}$-genera are obstructions to the existence of metrics satisfying certain curvature condition related to psc and provided some results in this direction. The higher $\widehat{A}$-genera, similar to the higher signature mentioned above, are characteristic numbers of the form $\langle \widehat{\mathcal{A}}(M)ch(V), [M] \rangle$, where $V \to M$ is a flat hermitian vector bundle. We would like to know what metric conclusions can be drawn from the vanishing of higher signatures.

References

[1] J. F. Adams. *Stable homotopy and generalised homology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1995. Reprint of the 1974 original.

[2] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. I. *Ann. of Math. (2)*, 86:374–407, 1967.

[3] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. II. Applications. *Ann. of Math. (2)*, 88:451–491, 1968.

[4] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69, 1975.

[5] M. F. Atiyah and I. M. Singer. The index of elliptic operators. I. *Ann. of Math. (2)*, 87:484–530, 1968.

[6] M. F. Atiyah and I. M. Singer. The index of elliptic operators. III. *Ann. of Math. (2)*, 87:546–604, 1968.

[7] Paul Baum and Ronald G. Douglas. K homology and index theory. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, volume 38 of *Proc. Sympos. Pure Math.*, pages 117–173. Amer. Math. Soc., Providence, R.I., 1982.

[8] R. Bott and A. Haefliger. On characteristic classes of Γ-foliations. *Bull. Amer. Math. Soc.*, 78:1039–1044, 1972.

[9] Raoul Bott. On a topological obstruction to integrability. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XVI, Berkeley, Calif., 1968)*, pages 127–131. Amer. Math. Soc., Providence, R.I., 1970.

[10] Raoul Bott. Lectures on characteristic classes and foliations. In *Lectures on algebraic and differential topology (Second Latin American School in Math., Mexico City, 1971)*, pages 1–94. Lecture Notes in Math., Vol. 279. 1972. Notes by Lawrence Conlon, with two appendices by J. Stasheff.

[11] Boris Botvinnik, Johannes Ebert, and Oscar Randal-Williams. Infinite loop spaces and positive scalar curvature. *arXiv e-prints*, page arXiv:1411.7408, Nov 2014.

[12] Thomas Church, Martin Crossley, and Jeffrey Giansiracusa. Invariance properties of Miller-Morita-Mumford characteristic numbers of fibre bundles. *Q. J. Math.*, 64(3):729–746, 2013.

[13] Johannes Ebert. Algebraic independence of generalized MMM-classes. *Algebr. Geom. Topol.*, 11(1):69–105, 2011.

[14] Johannes Ebert. A vanishing theorem for characteristic classes of odd-dimensional manifold bundles. *J. Reine Angew. Math.*, 684:1–29, 2013.

[15] Johannes Ebert. Index theory in spaces of manifolds. *Math. Ann.*, 374(1-2):931–962, 2019.

[16] Johannes Ebert and Oscar Randal-Williams. Infinite loop spaces and positive scalar curvature in the presence of a fundamental group. *arXiv e-prints*, page arXiv:1711.11363, Nov 2017.

[17] Sø ren Galatius, Ilya Grigoriev, and Oscar Randal-Williams. Tautological rings for high-dimensional manifolds. *Compos. Math.*, 153(4):851–866, 2017.

[18] Sø ren Galatius and Oscar Randal-Williams. Monoids of moduli spaces of manifolds. *Geom. Topol.*, 14(3):1243–1302, 2010.

[19] Sø ren Galatius and Oscar Randal-Williams. Detecting and realising characteristic classes of manifold bundles. In *Algebraic topology: applications and new directions*, volume 620 of *Contemp. Math.*, pages 99–110. Amer. Math. Soc., Providence, RI, 2014.

[20] Sø ren Galatius, Ulrike Tillmann, Ib Madsen, and Michael Weiss. The homotopy type of the cobordism category. *Acta Math.*, 202(2):195–239, 2009.

[21] Soren Galatius, Ib Madsen, and Ulrike Tillmann. Divisibility of the stable Miller-Morita-Mumford classes. *J. Amer. Math. Soc.*, 19(4):759–779, 2006.

[22] M. Gromov. Positive curvature, macroscopic dimension, spectral gaps and higher signatures. In *Functional analysis on the eve of the 21st century, Vol. II (New Brunswick, NJ, 1993)*, volume 132 of *Progr. Math.*, pages 1–213. Birkhäuser Boston, Boston, MA, 1996.

[23] Wolfgang Lück and Andrew Ranicki. Surgery obstructions of fibre bundles. *J. Pure Appl. Algebra*, 81(2):139–189, 1992.

[24] Ib Madsen and Michael Weiss. The stable moduli space of Riemann surfaces: Mumford's conjecture. *Ann. of Math. (2)*, 165(3):843–941, 2007.

[25] Yozô Matsushima. On Betti numbers of compact, locally sysmmetric Riemannian manifolds. *Osaka Math. J.*, 14:1–20, 1962.

[26] Werner Meyer. Die Signatur von lokalen Koeffizientensystemen und Faserbündeln. *Bonn. Math. Schr.*, (53):viii+59, 1972.

[27] John W. Milnor and James D. Stasheff. *Characteristic classes.* Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 76.

[28] Alexandr S. Mishchenko. K-theory over C^*-algebras. In *Geometry and topology of manifolds*, volume 76 of *Banach Center Publ.*, pages 245–266. Polish Acad. Sci. Inst. Math., Warsaw, 2007.

[29] Sam Nariman. Homological stability and stable moduli of flat manifold bundles. *Adv. Math.*, 320:1227–1268, 2017.

[30] Sam Nariman. Stable homology of surface diffeomorphism groups made discrete. *Geom. Topol.*, 21(5):3047–3092, 2017.

[31] S. P. Novikov. Algebraic construction and properties of Hermitian analogs of K-theory over rings with involution from the viewpoint of Hamiltonian formalism. Applications to differential topology and the theory of characteristic classes. I. II. *Izv. Akad. Nauk SSSR Ser. Mat.*, 34:253–288; ibid. 34 (1970), 475–500, 1970.

[32] Oscar Randal-Williams. Some phenomena in tautological rings of manifolds. *Selecta Math. (N.S.)*, 24(4):3835–3873, 2018.

[33] Jonathan Rosenberg. C^*-algebras, positive scalar curvature, and the Novikov conjecture. *Inst. Hautes Études Sci. Publ. Math.*, (58):197–212 (1984), 1983.

[34] Bena Tshishiku. Pontryagin classes of locally symmetric manifolds. *Algebr. Geom. Topol.*, 15(5):2709–2756, 2015.

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